

# A simplistic pedagogical formulation of the Maxwell-Boltzmann Thermal Speed Distribution using a relativistic framework

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## Abstract

A novel pedagogical technique is presented that can be used in the undergraduate (UG) class to formulate a relativistically extended Kinetic Theory of Gases and Maxwell-Boltzmann thermal speed distribution, while keeping the basic thermal symmetry arguments intact. The adopted framework can be used by students to understand the physics in a thermally governed system at high temperature and speeds, without having to indulge in high level tensor based mathematics.<sup>1</sup> Our approach will first recapitulate what is taught and known in the UG class and then present a methodology that will help students to understand and derive the physics of relativistic thermal systems. The methodology uses simple tools well known in the UG class and involves a component of computational techniques that can be used to involve students in this exercise. We also present towards the end the interesting implications of the relativistically extended distribution and compare it with Maxwell-Boltzmann results at various temperatures.

## I. INTRODUCTION

### A. Thermal Gas in the Undergraduate Class

The standard Kinetic Theory of Gases (KTG) as taught in a basic undergraduate (UG) course is a collision-based description of a classical ideal gas of particles,<sup>2</sup> with the walls of the container using a Newtonian framework and assumptions.<sup>3</sup> It results in expressions for the macroscopic properties of the gas (pressure, average kinetic energy  $\overline{K}$ , root mean square speed  $v_{\text{rms}}$  etc.) as a function of the absolute temperature  $T$  of the gas. The derivation of these results can be found in commonly used texts,<sup>4</sup> and yields the following major results for a classical gas of  $N$  particles each having a mass  $m$  with  $k_B$  being the Boltzmann constant:

$$\overline{v^2} = \frac{3k_B T}{m}, \quad (1)$$

$$\overline{K} = \frac{3Nk_B T}{2}. \quad (2)$$

A distribution function describes the probability of a particle's speed near a particular value as a function of the gas' absolute temperature, the mass of the particle and the value of speed under consideration. Random thermal motion dictates that both position and velocity space are uniformly distributed and the distribution function is stationary with time. A probability distribution function(PDF)  $f(.)$  of the speed of the particles in one direction must also give the directional probability in other independent directions as well. In the cartesian coordinate system, the probability of the speed of a particle to lie between  $v_k \rightarrow v_k + dv_k$  for  $k = \{x, y, z\}$  is given by  $f(v_x).f(v_y).f(v_z)dv_x dv_y dv_z$ . Since any direction is as good as the others, the resultant distribution function  $j(.)$  must only depend on the total speed  $v$  of the particle,

$$j(v^2) = j(v_x^2 + v_y^2 + v_z^2) = f(v_x).f(v_y).f(v_z). \quad (3)$$

The form of the directional function  $f(.)$  is clearly that of an exponential as seen by taking partial derivatives of Eq. (3) with respect to  $v_k$ , representing a Standard Probability Distribution function of the form,

$$f(v_k) = A e^{-B v_k^2}, \quad (4)$$

where  $A$  and  $B$  are constants depending on  $T$  and  $m$ . The Maxwell-Boltzmann Thermal speed distribution<sup>5</sup> (MB distribution) gives the probability  $P(v) dv$  of the particle speed to

lie within an elementary volume  $dv_x dv_y dv_z$ , or to lie between the speeds  $v$  and  $v + dv$  in the velocity space, centered on  $(v_x, v_y, v_z)$ , where  $dv_x dv_y dv_z \equiv 4\pi v^2 dv$ :

$$P(v)dv = f(v_x).f(v_y).f(v_z)dv_x dv_y dv_z = 4\pi A^3 v^2 e^{-Bv^2} dv. \quad (5)$$

The constants  $A$  and  $B$  in Eq. (5) are determined using the two integration conditions:

### 1. The Classical All Particle Condition

Classically and non-relativistically, the speed  $v$  of all particles follows  $0 \leq v < \infty$ , hence the PDF  $P(v)$  of Eq. (5) can be normalized as:

$$\int_0^\infty P(v) dv = \int_0^\infty 4\pi A^3 v^2 e^{-Bv^2} dv = 1. \quad (6)$$

### 2. The Classical Kinetic Theory of Gases $v_{rms}$ Result

Relating Eq. (1) with the probability distribution function of Eq. (5), we get,

$$\overline{v^2} = \int_0^\infty v^2 P(v) dv = \int_0^\infty 4\pi A^3 v^4 e^{-Bv^2} dv = \frac{3kT}{m}. \quad (7)$$

Thus,

$$A = \left(\frac{m}{2\pi kT}\right)^{1/2} \quad B = \left(\frac{m}{2kT}\right). \quad (8)$$

## B. Need and Motivation to extend the Maxwell-Boltzmann Distribution

The following points on the formulation of the MB distribution yield discrepancies in regard to the Special Theory of Relativity:

1. The classical KTG is based on Newtonian mechanics and does not consider the effect of Special Relativity. The Normal probability distribution that the MB distribution uses in Eq. (5) is a gaussian and the variable, the speed of the particle  $v$ , extends till infinity. Classical MB distribution assumes that  $v \in [0, \infty)$  as in the normalization scheme of Eq. (6), but Einstein's Special Theory of Relativity<sup>6</sup> limits  $v$  to  $c$ : the speed of light in free space. MB distribution at any  $T$  will always predict a non-zero fraction of particles to have speeds greater than  $c$ , which is physically forbidden and leads to inaccuracies in the distribution, particularly at high temperatures.

2. As the particle gas approaches high temperatures, more particles will have  $v \rightarrow c$  and hence relativistic effects cannot be ignored. Thus, we cannot use  $\overline{K} = (m\overline{v^2}/2)$  as is implied by Eq (1). Rather, using Special Theory,  $\overline{K} = (\overline{\gamma} - 1)mc^2$  where  $\gamma = (1 - v^2/c^2)^{-1/2}$  is the Lorentz factor. The basic result of the KTG in Eq. (1) can be extended under the constraints of the Special Theory of Relativity as done in the next section.

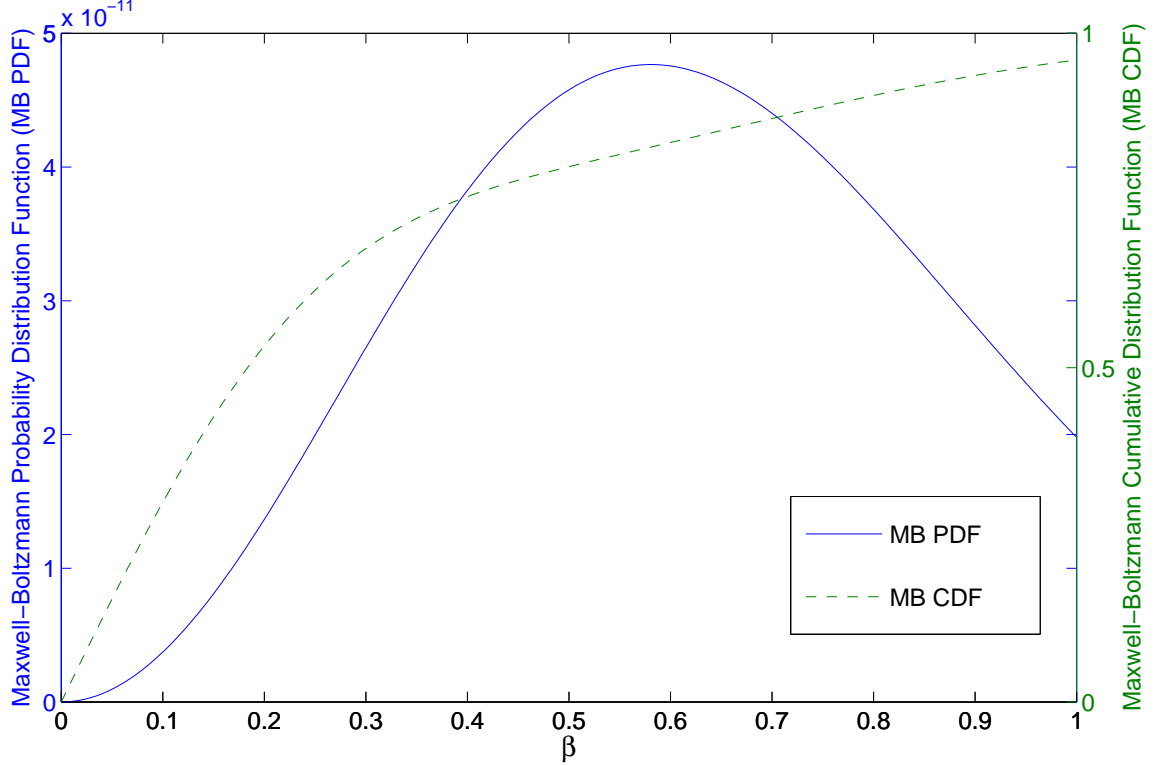


FIG. 1. MB PDF (Solid Curve) and CDF (Broken Curve) for a classical electron gas at  $T = 10^9$  K

Astrophysical systems, e.g. the Intra cluster medium of galaxy clusters, can reach temperatures as high as  $10^8$  K,<sup>7</sup> which involve relativistic speeds and describing processes like Thermal Bremsstrahlung Emission<sup>8</sup> using MB distributions will lead to errors as depicted with the help of Fig. 1. It depicts the MB distribution for a classical electron gas at  $T = 10^9$  K with  $v_{\text{most probable}} = 0.58 c$  and shows that the Cumulative Distribution Function (CDF)  $\neq 1$  as  $v \rightarrow c$  and predicts about 3.2 % particles to have speeds beyond  $c$ . The error in the MB distribution is beyond negligence and is not an accurate description of the system. MB dis-

tributions are used extensively in the UG curriculum to describe different thermal systems, whose speeds can be relativistic as shown above and to allow for deeper understanding of the system at student level, we are motivated to extend the classical KTG and MB distribution into the relativistic regime by incorporating the postulates of Special Theory of Relativity.

## II. PEDAGOGICAL EXTENSION OF KTG INTO THE RELATIVISTIC REGIME

We have achieved a novel approach to extend the KTG into the relativistic regime using Four Vectors<sup>9</sup>, a standard tool of the undergraduate physics class. This approach allows us to understand the physics behind the physical origin of the gas' macroscopic properties on similar lines of the standard KTG derivation using a collision based approach, under the constraints of Special Theory of Relativity. Previous works by Synge and Jüttner,<sup>110</sup> have developed a relativistic thermal speed distribution with the usage of tensor calculus, a mathematical tool not available with all UG students. Our methodology using Four Vectors uses minimal and elementary mathematics but offers insight into the physics of the problem.

From an inertial frame of reference  $S$ , we observe a box of volume  $V = l^3$  containing  $N$  classical particles with all standard KTG assumptions valid except that relativistic effects can be ignored. Let the  $i^{\text{th}}$  particle in the box be described by the position four vector  $X \equiv (x^0, x, y, z)$ . Let  $\Delta\tau$  be the proper time between events as measured in the particle's rest frame  $S'_i$  and  $\Delta t$  be the improper time between events as measured in the frame  $S$  which are related by the Lorentz factor as:

$$\frac{d\tau}{dt} = (1 - \frac{v^2}{c^2})^{-1/2} = \gamma. \quad (9)$$

The corresponding acceleration four vector can be written in a form,<sup>11</sup>

$$A_i \equiv (\dot{\gamma}\gamma c, \gamma^2 \vec{a} + \dot{\gamma}\gamma \vec{v}), \quad (10)$$

where  $\vec{v}$  is the spatial velocity of the particle,  $\vec{a} = d\vec{v}/dt$  and  $\dot{\gamma} = \gamma^3(\vec{a} \cdot \vec{v})/c^2$ .

We first model the particle-wall elastic collision in one dimension in the frame  $S$  and then generalize the results to three dimensions. In a one dimensional analysis along an axis perpendicular to a particular wall of the container, say  $x$ -axis:  $\vec{v} = v_x \hat{x}$  and the acceleration  $\vec{a} = d\vec{v}/dt = \Delta\vec{v}/\Delta t$ . A similar approach as evolved in the Standard Newtonian KTG can be extended here such that on each collision with the particular wall of the container, we

have the following:

$$\Delta \vec{v} = 2v_x \hat{x}, \quad (11)$$

$$\Delta t = \frac{2l}{v_x}, \quad (12)$$

$$\vec{a} = \frac{v^2}{l} \hat{x} = \frac{v_x^2}{l} \hat{x}, \quad (13)$$

$$\vec{a} \cdot \vec{v} = \frac{v_x^3}{l}. \quad (14)$$

Thus the spatial component of the acceleration four vector takes the form of,

$$A_{\text{spatial},i,1D} = \frac{(v^2)(\gamma_x^2)}{l} + \frac{(v^4)(\gamma_x^4)}{lc^2} = \frac{(v^2)(\gamma_x^2)}{l} \left[ 1 + \frac{(v^2)(\gamma_x^2)}{c^2} \right], \quad (15)$$

where  $\gamma_x = (1 - v_x^2/c^2)^{-1/2}$  is the Lorentz factor assigned in one dimension and since  $[1 + (v^2(\gamma_x^2)/c^2)] = (\gamma_x^2)$ , hence we get:

$$A_{\text{spatial},i,1D} = \frac{(v_x^2)(\gamma_x^4)}{l}. \quad (16)$$

The spatial part of the force four vector can be written as  $F_{\text{spatial},i,1D} = m A_{\text{spatial},i,1D}$  and macroscopic gas properties can be derived as follows. The pressure due to the  $i^{\text{th}}$  particle is

$$P_{i,1D} = \frac{(mv_x^2)(\gamma_x^4)}{V}. \quad (17)$$

The total pressure of the gas can be found by summing  $P_{i,1D}$  in Eq. (17) over all the  $N$  particles in the box

$$P_{\text{total}} = \sum_{i=1}^N P_{i,1D} = \sum_{i=1}^N \frac{Nmc^2}{V} \overline{(\beta_x^2)(\gamma_x^4)}, \quad (18)$$

where  $\beta = v/c$  and dimension-specific beta  $\beta_k = v_k/c$  for  $k = \{x, y, z\}$ .

The particles of the gas are in constant random motion under thermal equilibrium which provides that all directions away from the walls of the container are identical in all physical respects and no particular direction is preferred over the other. The symmetry in all the three cartesian directions due to thermal equilibrium of the gas leads to the conclusion that the average contribution of each cartesian direction component is the same,

$$\bar{\beta}_x^2 = \bar{\beta}_y^2 = \bar{\beta}_z^2 = \frac{\bar{\beta}^2}{3}, \quad (19)$$

where  $\beta_x, \beta_y, \beta_z$  are related to the beta factor  $\beta$  of the particle as,  $\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2$ . Mathematical analysis using infinite geometric series helps alternatively express Eq. (18) by introducing average of powers of  $\beta$  using Eq. (19) as:

$$P_{\text{total}} = \frac{3Nmc^2}{V} \overline{(\beta^2)(\theta^4)}, \quad (20)$$

where

$$\theta^2 = \frac{1}{3 - \beta^2} . \quad (21)$$

As with the classical KTG, we assume that the gas particles behave ideally and follow the Ideal Gas Equation ( $PV = Nk_B T$ ) to generate the final Relativistic KTG postulate

$$\overline{(\beta^2)(\theta^4)} = \frac{k_B T}{3mc^2} . \quad (22)$$

Equation (22) is the relativistically extended version of the classical KTG postulate as in Eq. (1) by incorporating constraints of Special Relativity and can supersede the classical KTG postulate.

### III. FORMULATING A RELATIVISTIC DISTRIBUTION FUNCTION

The underlying symmetry arguments in the MB distribution of uniform occupation of position and velocity space and equivalence among different directions as stated in §I (A) are valid and are the defining criteria for random motion in thermal equilibrium. We can formulate the relativistically extended (RE) distribution function by incorporating changes in Eq. (5) compliant with Special Relativity. Let  $J(v)dv$  be the RE distribution function that has all the features of a thermal distribution. Postulates of Special Relativity will dictate that  $J(0) = J(v \geq c) = 0$  because no particle can travel with the vacuum speed of light or beyond. Eq. (3) will hold for the functional form of  $J(v)dv$  and the directional distribution function  $f_{\text{RE}}(.)$  can be assumed on similar lines of §I (A) to be  $f_{\text{RE}}(v_k) = P e^{[-Rv_k^2 \gamma^2]}$  with  $P$  and  $R$  as constants.

The resultant RE distribution function in velocity space centered on  $(v_x, v_y, v_z)$  or equivalently in  $\beta$  space centered on  $(\beta_x, \beta_y, \beta_z)$  is, respectively,

$$J(v)dv = 4\pi P^3 v^2 e^{(-Rv^2 \gamma^2)} dv , \quad (23)$$

$$J(\beta)d\beta = 4\pi X^3 \beta^2 e^{(-Y\beta^2 \gamma^2)} d\beta , \quad (24)$$

and the constants in these equations being related as  $X = Pc$  and  $Y = Rc^2$

It is important to mention at this stage that  $J(\beta)d\beta$  is not strictly a Normal Probability Distribution<sup>12</sup> because the argument of the exponential in Eq. (24) does not vary as  $\beta^2$  but

as  $\beta^2\gamma^2$  i.e.  $(\beta^2/1 - \beta^2)$ . Hence, Eq. (24) is a Probability Representative Function (PRF) and not strictly a PDF since it predicts a number proportional to the probability. The constants  $X$  and  $Y$  in Eq. (24) are dependant on  $T$  and  $m$ , and can be determined using the following two conditions:

### 1. The Relativistic All Particle Condition

As constrained by the Special Theory of Relativity, the speed  $v$  of any particle follows  $0 \leq v < c$ , hence the PRF  $J(\beta)d\beta$  of Eq. (24) can be normalized to include all the  $N$  gas particles as:

$$\int_0^1 J(\beta) d\beta = \int_0^1 4\pi X^3 \beta^2 e^{-Y\beta^2\gamma^2} d\beta = 1. \quad (25)$$

### 2. The Relativistically Extended KTG Result

Relating Eq. (22) and (21) with the PRF of Eq. (24), one gets:

$$\overline{\beta^2\theta^4} = \int_0^1 \beta^2\theta^4 J(\beta) d\beta = \int_0^1 4\pi X^3 \frac{\beta^4}{(3 - \beta^2)^2} e^{-Y\beta^2\gamma^2} d\beta = \frac{kT}{3mc^2}. \quad (26)$$

The integral conditions of Eq. (25) and (26) are not solvable in closed analytic form. The computational facility Wolfram Alpha<sup>13</sup> available was used to numerically solve these for a classical ideal gas of electrons by feeding a 40-value long array of constant  $Y$  to generate values of corresponding  $X$  and  $T$ . The Curve Fitting Tool of MATLAB<sup>14</sup> can be used to plot the temperature dependance of  $X$  and  $Y$  (Fig. 2) and apply a best fit for the data wherever possible. A linear polynomial best fit the correlation of  $\ln(X) - \ln(T)$  and the  $T$  dependance of  $X$  is

$$\ln X = 10.4 - (0.5063) \ln T. \quad (27)$$

## IV. FEATURES OF THE RELATIVISTICALLY EXTENDED DISTRIBUTION

### A. The Most Probable Speed Analysis

The most probable (mp) speed of the particles in the MB distribution is  $v_{\text{mp}} = (2kT/m)^{1/2}$ . A similar expression using the RE distribution can be found when the PRF of Eq. (24) is maximum, which gives

$$\frac{\beta_{\text{mp}}^2}{(1 - \beta_{\text{mp}}^2)^2} = \frac{1}{Y}, \quad (28)$$



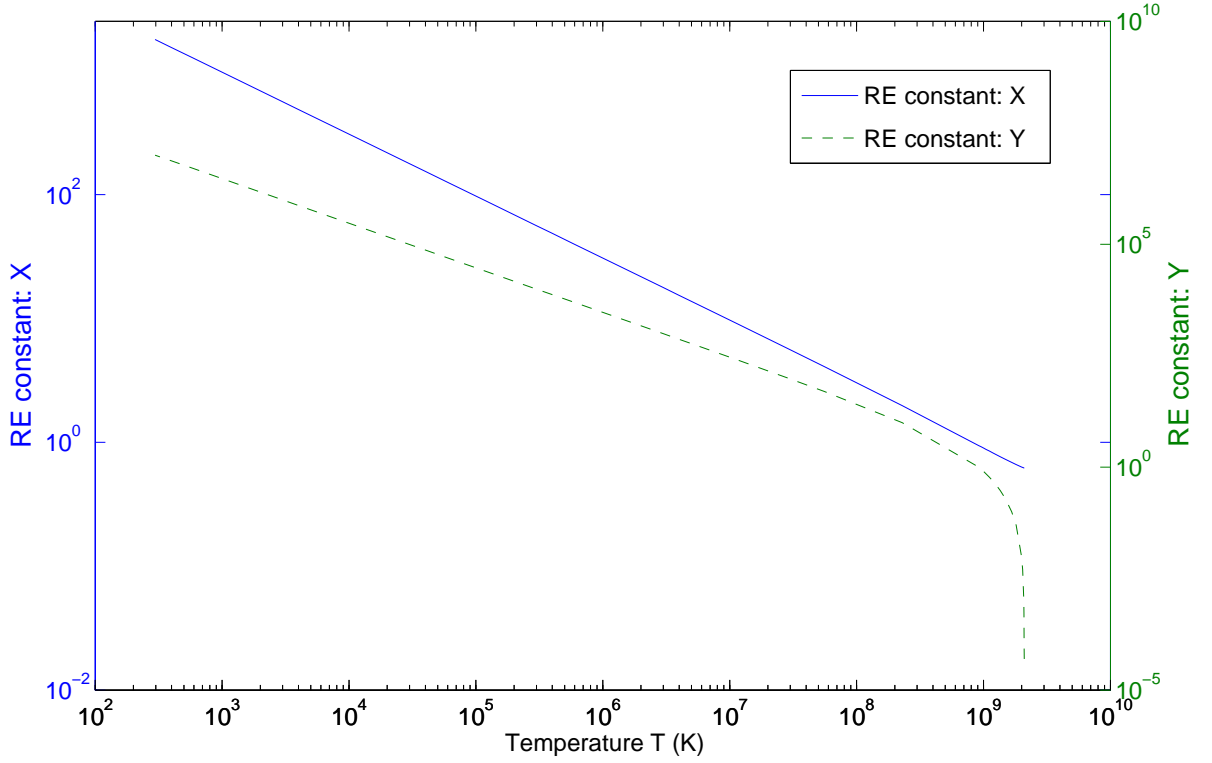


FIG. 2. Temperature dependance of constants X (Solid Curve) and Y (Broken Curve) of Eq. (24)

which can be solved using a quadratic analysis of roots. In the non-relativistic limit  $v \ll c$  so that  $\gamma \rightarrow 1$ ,  $\beta_{\text{mp}}^2 \approx (1/Y) = (1/Rc^2)$  from Eq. (24) and (23), which is equivalent to  $v_{\text{mp}}^2 \approx (1/R)$ . The constant  $R$  in the MB PDF of Eq. (5) is  $\frac{m}{2kT}$  which gives us,

$$v_{\text{mp}, v \ll c} = \left( \frac{2kT}{m} \right)^{1/2}, \quad (29)$$

and hence in the non-relativistic regime, the results for most probable speed using the RE distribution match the MB results.

### B. The nature of temperature dependance of constant Y

The argument of the exponential in the PRF of Eq. (23) must be unit-less, hence  $R$  will have the units of ( $\text{sec}^2/\text{m}^2$ ) or equivalently ( $\text{kg}/\text{joule}$ ) i.e inverse of specific work.  $R$  in Eq. (23) is inversely proportional to the work or energy required to heat the gas to the temperature  $T$ . The temperature dependance of  $Y$  as in Fig. 2 is analogous to the x-axis reflected plot of total energy  $E = (mc^2/\sqrt{1-\beta^2})$  of a particle with it's  $\beta$  parameter.

As the temperature and the required energy of the gas to accomplish this increases there is decrease in  $Y$ . After a certain  $T$  range, any further increase in  $T$  requires an uncontrollable increase in energy input and a corresponding uncontrollable decrease in  $Y$ . It is clear that a high  $T$  involves relativistic speeds and nature demands lot of energy to sustain such high speeds and temperatures.

## V. A COMPARISON STUDY

We now compare results of the MB and RE distributions at various temperatures and show that results of MB distribution for  $T \gtrsim 5 \times 10^8 \text{K}$  are inaccurate beyond negligence and the RE distribution results are physically convincing and can be a good approximation to the real system. As stated in §III, Eq. (24) is a PRF and predicts a number proportional to the probability. The MB and RE distributions can be compared at a given  $T$  by a parameter ( $r$ ) at various speeds from Eq. (5) and Eq. (23).

$$\text{Ratio}(r) = \frac{P(v)}{J(v)} . \quad (30)$$

We will use  $r_{\max}$  to represent the Ratio ( $r$ ) at the highest plotted speed and  $r_{\min}$  will represent the Ratio ( $r$ ) at the lowest plotted speed and  $\Delta r = r_{\max} - r_{\min}$ , an average estimate of the scatter between the two functions at a given  $T$ .

Table I chalks out a comparison (Fig. 3 and 4) between the MB and RE distribution

TABLE I. Comparing MB and RE distribution functions at two different temperatures

$T = 2.96 \times 10^5 \text{ K}$	$T = 9.18 \times 10^8 \text{ K}$
$r_{\max} = 3.52 \times 10^{-11}$	$r_{\max} = 4.31 \times 10^9$
$r_{\min} = 3.53 \times 10^{-11}$	$r_{\min} = 4.16 \times 10^{-11}$
$\Delta r = 1.86 \times 10^{-12}$	$\Delta r = 4.31 \times 10^9$
$v_{(\text{mp-MB})} = 0.0099984c$	$v_{(\text{mp-MB})} = 0.5565418c$
$v_{(\text{mp-RE})} = 0.009999c$	$v_{(\text{mp-RE})} = 0.618034c$
$100 \times \int_c^\infty P(v) dv = 0$	$100 \times \int_c^\infty P(v) dv = 3.2\%$
$\int_c^\infty J(v) dv = 0$	$\int_c^\infty J(v) dv = 0$

functions at two different temperatures by comparing values of various parameters,

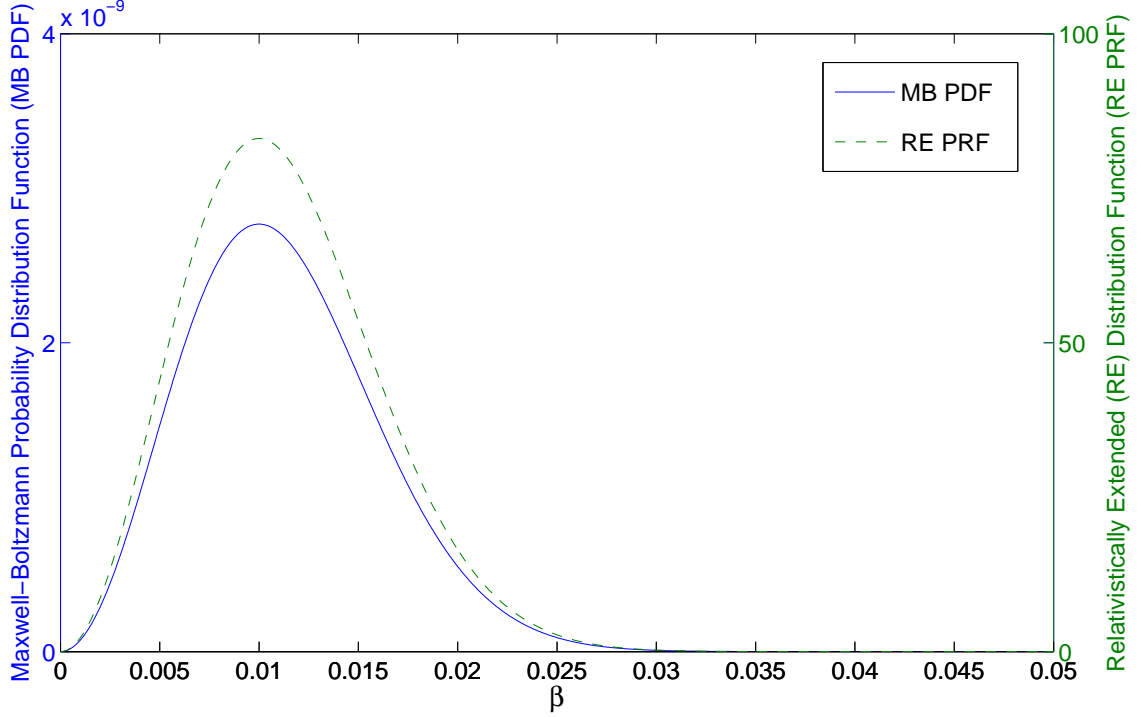


FIG. 3. MB PDF (Solid Curve) and Relativistically Extended (RE) PRF (Broken Curve) for  $T = 2.96 \times 10^5$  K

representative of each distribution. As evident from the comparison table, for  $T \lesssim 5 \times 10^8$  K, MB and RE distributions match very well with the shape of the distribution function preserved and the scatter parameter  $r$  between the two is of the order of  $10^{-12}$  or less. RE distribution converges to Maxwellian results at low speeds and low temperatures.

Figures 3 and 4 plot MB and RE distribution functions at the two comparison temperatures. They clearly show that for higher  $T$ , the shape of the two functions varies significantly, with MB predicting about 3.2% particles beyond  $c$ , while the RE function is a good description of the system under the constraints of special relativity. Figure 5 shows the variation of the Ratio  $r$  with  $\beta$  for  $T = 2.96 \times 10^5$  K. The scatter in the two distributions is extremely large as  $v \rightarrow c$  but it is important to notice that at all temperature regimes, RE results converge to MB at low speeds. Overall, the RE and MB distributions diverge at high  $T$  and high  $v$  and the RE distribution favors higher speeds compared to MB, as shown by the  $v_{\text{mp}}$ .

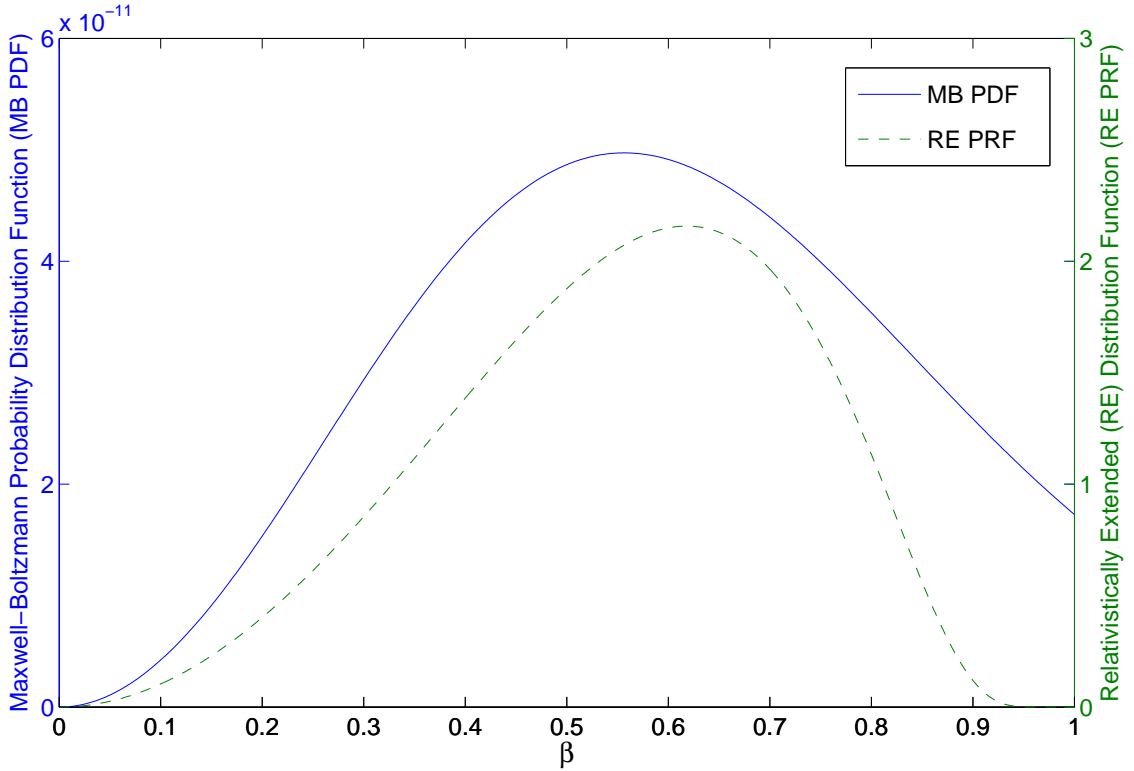


FIG. 4. MB PDF (Solid Curve) and Relativistically Extended (RE) PRF (Broken Curve) for  $T = 9.18 \times 10^8$  K

## VI. CONCLUSION

In our analysis of including Special Relativity in the KTG and MB distribution, we have presented a pedagogically useful method to introduce relativistic constraints that preserve the basic collision and thermal framework that is taught as the standard approach to this subject. The method of Four Vectors and particle-wall collisions along with the usage of computational techniques to solve integral conditions can be very effectively used in the UG classroom paradigm to help students appreciate the interlink between different areas of Physics and themselves try and formulate it. The ideas presented here give a reasonable description of the physical system but it is important to note that this approach is based on incorporation of Special Relativity in a previously accepted methodology and the formulation of the RE function involves assuming a logically sound function, extrapolating from the MB framework. Thus, we suggest that due care must be taken while adopting our approach which we feel is a good pedagogical tool in the UG classroom.

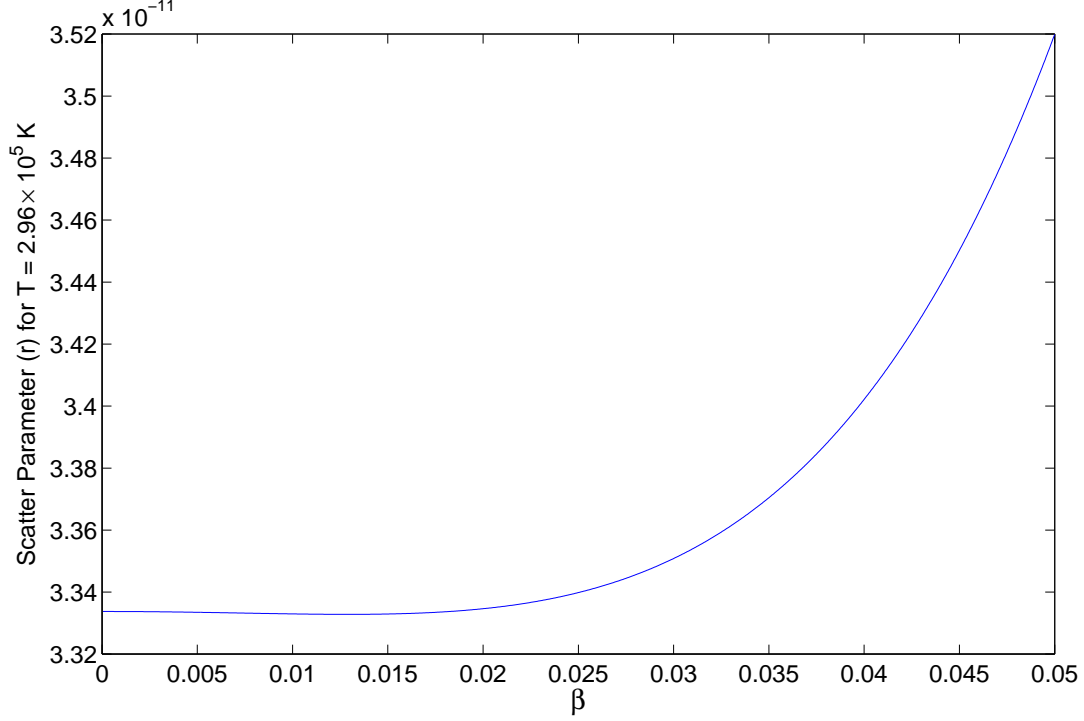


FIG. 5. Scatter Ratio "r" for  $T = 2.96 \times 10^5$  K

## ACKNOWLEDGMENTS

I am most grateful to Prof. Joydeep Bagchi and Dr. Surajit Paul of the Inter University Center for Astronomy and Astrophysics (IUCAA), Pune, India for their support, guidance and motivation. I am also grateful to Prof. Tashi Nautiyal of the Indian Institute of Technology Roorkee, Roorkee, India for kindly consenting to proofread the manuscript.

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